Non-tangential, radial and stochastic asymptotic properties of harmonic functions on trees *†

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Abstract

For a harmonic function on a tree with random walk whose transition probabilities are bounded between two constants in $(0, 1/2)$, it is known that the radial and stochastic properties of convergence, boundedness and finiteness of energy are all a.s. equivalent. We prove here that the analogous non-tangential properties are a.e. equivalent to the above ones.

We are interested in the comparison between some non-tangential asymptotic properties of harmonic functions on a tree and the corresponding radial properties, using analogous stochastic ones. We proved in a previous work [6], under a reasonable uniformity hypothesis, the almost sure equivalence between different radial and stochastic properties: convergence, boundedness and finiteness of the energy. The probabilistic-geometric methods, adaptated from those we used in the setting of manifolds of negative curvature [5], were flexible and presumed to extend to the non-tangential case for trees.

A recent article [2] shows by combinatorial methods the equivalence of the three non-tangential corresponding properties in the particular case of homogeneous trees. It seems to be time to show explicitely that our methods give in a swift way the non-tangential results for general trees satisfying the uniformity hypothesis above.

We use our previous results to compare the non-tangential notions with the radial and stochastic ones: we prove on one hand that the stochastic convergence implies the non-tangential convergence in the section 3 and on the other hand that the non-tangential boundedness implies almost surely the finiteness of the non-tangential energy in the section 4. The notations are fixed in the section 1 and our main result is stated in the section 2.

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1 Setting

Let us briefly fix the notations (for details see [6]). We consider a *tree* (S, A) i.e. a non-oriented, locally finite, connected and simply connected graph with *vertices* in S and *edges* in A. We will use the usual notions of *path*, *distance* and *geodesic path* and note $x \sim y$ iff $(x, y) \in A$.

We also consider a transient random walk $(X_n)_n$ on S such that the transition probability $p(x, y) > 0$ iff $x \sim y$. Denote by P_x the distribution of the walk starting from x and by $p_n(x, y)$ the probability $P_x[X_n = y]$ of reaching y from x in n steps.

The *Green function* $G(x, y) = \sum_{n=0}^{\infty} p_n(x, y)$ is finite by transience. Denote by $H(x, y)$ the probability of reaching y starting from x. If z is on the geodesic path $[x, y]$, the simple connectivity implies

$$H(x, y) = H(x, z)H(z, y) \quad \text{and} \quad G(x, y) = H(x, z)G(z, y). \tag{1}$$

If $U \subset S$, the *Green function of* U, defined on $U \times U$, is the expectation of the number of times the walk starting from x hits y before exiting U.

The *Laplacian* of a function f on S is $\Delta f(x) = E_x[f(X_1)] - f(x)$. The function f is *harmonic* if $\Delta f = 0$.

Let u be a fixed harmonic function. The *stochastic energy* of u is $J^*(u) = \sum_{k=0}^{\infty} \left(\Delta u^2\right)(X_k)$ (non-negative terms). The events $\mathcal{L}^{**}$, $\mathcal{N}^{**}$ and $\mathcal{J}^{**}$ are defined respectively by the convergence of $(u(X_n))_n$, its boundedness and the finiteness of the stochastic energy. The Martingale theorem implies $\mathcal{J}^{**} \overset{\sim}{\subset} \mathcal{L}^{**}$ (P_x-almost sure inclusion) [6]. It is known since P. Cartier [3] that geometric and Martin compactifications agree and the random walk converges almost surely to a point of the boundary ∂S. The exit law starting from x is the harmonic measure μ_x and $\mu = (\mu_x)_x$ is a familly of equivalent measures. Conditioning by Doob's method of h-processes gives probabilities P_x^θ (ending at θ). Asymptotic events verify 0–1 law and we define sets $\mathcal{L}^* = \left\{\theta \in \partial S | P_x^\theta(\mathcal{L}^{**}) = 1\right\}$, $\mathcal{N}^* = \left\{\theta \in \partial S | P_x^\theta(\mathcal{N}^{**}) = 1\right\}$, $\mathcal{J}^* = \left\{\theta \in \partial S | P_x^\theta(\mathcal{J}^{**}) = 1\right\}$, which determine stochastic notions of convergence, boundedness and finiteness of the energy at $\theta \in \partial S$. For $\theta \in \mathcal{L}^*$, $\lim u(X_n)$ is P_x^θ-a.s. constant (independent from x) and called the *stochastic limit* at θ.

Fix a base point o. For $\theta \in \partial S$, γ_θ is the geodesic ray from o to θ and for $c \in \mathbf{N}$, $\Gamma_c^\theta = \{y \in S | d(y, \gamma_\theta) \leq c\}$ is a *non-tangential tube*. Let u be a harmonic function. For $c \in \mathbf{N}$, its *c-non-tangential energy* at θ is $J_c^\theta(u) = \sum_{y \in \Gamma_c^\theta} \Delta u^2(y)$ and its *radial energy* at θ is $J^\theta(u) = J_0^\theta(u) = \sum_{k=0}^{\infty} \Delta u^2(\gamma_\theta(k))$. There is *radial* convergence, boundedness or finiteness of the energy depending wether $(u(\gamma_\theta(n)))_n$ converges, is bounded or has finite radial energy. There is *non-tangential* convergence of u at θ if for all $c \in \mathbf{N}$, $u(y)$ has a limit when y goes to θ staying in Γ_c^θ. There is *non-tangential* boundedness (resp. finiteness of the energy) if for all $c \in \mathbf{N}$, u is bounded on Γ_c^θ (resp. $J_c^\theta(u) < +\infty$).

2 Main result

We now suppose $(\mathcal{H})$: $\exists\varepsilon > 0, \exists\eta > 0, \forall x \sim y, \varepsilon \leq p(x, y) \leq \frac{1}{2} - \eta$, a discrete analogue of the pinched curvature for manifolds. It also forces at least three neighbors for each vertex, and ensures transience. We proved in [6]:

Theorem 2.1 *For a harmonic function u on a tree with random walk satisfying $(\mathcal{H})$, the notions of radial convergence, radial boundedness, radial finiteness of the energy, stochastic convergence, stochastic boundedness, stochastic finiteness of the energy, are μ-almost equivalent.*

We prove here the following theorem:

Theorem 2.2 *Under the same hypotheses, the notions of non-tangential convergence, non-tangential boundedness and non-tangential finiteness of the energy are μ-almost equivalent to the notions above.*

Considering the trivial implications, it is sufficient to prove that stochastic convergence implies non-tangential convergence and non-tangential boundedness implies almost surely non-tangential finiteness of the energy.

3 Stochastic implies NT convergence

The first implication needs the following lemma due to A. Ancona in a general setting [1], but easily proved here by simple connectivity:

Lemma 3.1 *If $(x_n)_n$ is a sequence converging non-tangentially to $\theta \in \partial S$, the walk hits P_o^θ-a.s. infinitely many x_n.*

Let us see how this lemma helps. Assume that the harmonic function u has a stochastic limit $l \in \mathbf{R}$ at θ but does not converge non-tangentially towards l at θ. There exists $\delta > 0$ and a sequence $(x_n)_n$ converging non-tangentially to θ such that $|u(x_n) - l| \geq \delta$ for all n. As the random walk $(X_k)_k$ hits P_o^θ-a.s. infinitely many x_n by the lemma, one can extract a subsequence $(X_{k_j})_j$ such that $|u(X_{k_j}) - l| \geq \delta$ for all j. Hence, P_o^θ-almost surely, the function u does not converge towards l along $(X_k)_k$ which leads to a contradiction.

Le us now prove the lemma. Recall that the principle of the method of Doob's h-processes is to consider a new Markov chain defined by $p^\theta(x, y) = \frac{K_\theta(y)}{K_\theta(x)} p(x, y)$ where the Martin kernel $K_\theta(x)$ is defined as $\lim_{y\to\theta} \frac{G(x,y)}{G(o,y)}$ (see for example [4]). This formula leads to analogous fomulae for the p_n^θ and the associated functions H^θ and G^θ. Consider for a fixed n the projection y_n of x_n on the geodesic ray γ_θ (see [6]). As the random walk starting from o and conditioned to end at θ hits almost surely y_n due to the tree structure, the strong Markov property gives $H^\theta(o, x_n) = H^\theta(y_n, x_n) = \frac{K_\theta(x_n)}{K_\theta(y_n)} H(y_n, x_n)$. By definition of the Martin kernel, $\frac{K_\theta(x_n)}{K_\theta(y_n)} = \lim_{y\to\theta} \frac{G(x_n,y)}{G(y_n,y)}$ and $G(x_n, y) = H(x_n, y_n)G(y_n, y)$ as soon as $y_n \in [x_n, y]$, so $H^\theta(o, x_n) = H(x_n, y_n)H(y_n, x_n)$. The distance between x_n and

y_n is bounded as $(x_n)_n$ converges non-tangentially to θ, hence the last product is bounded from below by a constant $C > 0$ using $(\mathcal{H})$. By Fatou's lemma, the probability conditioned to end at θ of hitting infinitely many x_n is not smaller than C and the asymptotic 0-1 law ensures that it equals 1, which completes the lemma's proof.

4 NT boundedness implies finite NT energy

Denoting $\mathcal{N}_c = \{\theta \in \partial S | \sup_{\Gamma_c^\theta} |u| < +\infty\}$ and $\mathcal{J}_c = \{\theta \in \partial S | J_c^\theta(u) < +\infty\}$, we will show that for all $c \in \mathbf{N}$, $\mathcal{N}_{c+1} \overset{\sim}{\subset} \mathcal{J}_c$, which will give the wanted result by monotonous intersection. Let us write $\mathcal{N}_{c+1} = \bigcup_{N \in \mathbf{N}} \mathcal{N}_{c+1}^N$, where

$$\mathcal{N}_{c+1}^N = \left\{ \theta \in \partial S \;\middle|\; \sup_{\Gamma_{c+1}^\theta} |u| \leq N \right\}.$$

By countability it is sufficient to prove that for all N, $\mathcal{N}_{c+1}^N \overset{\sim}{\subset} \mathcal{J}_c$. Let us fix $N \in \mathbf{N}$. Denote $\Gamma = \bigcup_{\theta \in \mathcal{N}_{c+1}^N} \Gamma_c^\theta$ and τ the exit time from Γ. As

$$M_n = u^2(X_n) - \sum_{k=0}^{n-1} \Delta u^2(X_k)$$

is a martingale (see [6]), Doob's stopping time theorem for the bounded exit time $\tau \wedge n$ gives $E_o\left[M_{\tau\wedge n}\right] = E_o[M_0] = u^2(o) \geq 0$, hence

$$E_o\left[\sum_{k=0}^{\tau\wedge n-1} \Delta u^2(X_k)\right] \leq E_o\left[u^2(X_{\tau\wedge n})\right].$$

As $X_{\tau\wedge n}$ is at distance at most 1 from Γ, it lies in a tube Γ_{c+1}^θ where $\theta \in \mathcal{N}_{c+1}^N$ and $|u(X_{\tau\wedge n})| \leq N$. When n goes to ∞, monotonous convergence ($\Delta u^2 \geq 0$) and the desintegration formula (see [6]) give then, for μ-almost all $\theta \in \partial S$,

$$E_o^\theta\left[\sum_{k=0}^{\tau-1} \Delta u^2(X_k)\right] < +\infty.$$

Let us use a conditioned version of formula 2 from [6], which will be proved later :

Lemma 4.1 *For a function $\varphi \geq 0$ on Γ and τ the exit time of Γ,*

$$E_o^\theta\left[\sum_{k=0}^{\tau-1} \varphi(X_k)\right] = \sum_{y\in\Gamma} \varphi(y) G_\Gamma(o,y) K_\theta(y).$$

This lemma implies that for μ-almost all $\theta \in \partial S$, $\sum_{y\in\Gamma} \Delta u^2(y) G_\Gamma(o,y) K_\theta(y)$ is finite. In order to get an energy, we will show that $G_\Gamma(o,y)K_\theta(y)$ is bounded from below using the two following lemmas. The first one is due to A. Ancona [1] but has a very simple proof in the present context of trees. The second one enables comparison between G_Γ and G.

Lemma 4.2 $\forall c \in \mathbf{N}, \exists \alpha > 0, \forall \theta \in \partial S, \forall y \in \Gamma_c^\theta, G(o,y)K_\theta(y) \geq \alpha.$

Lemma 4.3 *For $U \subset S$ containing Γ_c^θ and τ the exit time of U,*

$$\lim_{y\in\Gamma_c^\theta, y\to\theta} \frac{G_U(o,y)}{G(o,y)} = P_o^\theta[\tau = +\infty].$$

By lemma 4.2, for μ-almost all $\theta \in \mathcal{N}_{c+1}^N$,

$$\sum_{y\in\Gamma_c^\theta} \Delta u^2(y) \frac{G_\Gamma(o,y)}{G(o,y)} < +\infty.$$

If we show that for μ-almost all $\theta \in \mathcal{N}_{c+1}^N$, $P_o^\theta[\tau = +\infty] > 0$, lemma 4.3 gives $\mathcal{N}_{c+1}^N \overset{\sim}{\subset} \mathcal{J}_c$. The proof of that fact is the same as in the analogous radial proof [6] which completes the theorem's proof.

Let us now prove the lemmas. Concerning lemma 4.1, using Fubini,

$$E_o^\theta\left[\sum_{k=0}^{\tau-1} \varphi(X_k)\right] = \sum_{k=0}^{\infty} E_o^\theta\left[\varphi(X_k)\mathbf{1}_{(k<\tau)}\right].$$

The random variable $\varphi(X_k)\mathbf{1}_{(k<\tau)}$ being measurable with respect to the σ-algebra generated by $(X_i)_{i\leq k}$ (see [6]) and using formula 2 from [6], the expectation above equals

$$\sum_{k=0}^{\infty} E_o\left[\varphi(X_k)\mathbf{1}_{(k<\tau)}K_\theta(X_k)\right] = E_o\left[\sum_{k=0}^{\infty} \varphi(X_k)\mathbf{1}_{(k<\tau)}K_\theta(X_k)\right]$$
$$= \sum_{y\in\Gamma} \varphi(y) G_\Gamma(o,y) K_\theta(y),$$

which finishes the proof of lemma 4.1.

Let us prove lemma 4.2. Denote $\pi(y)$ the projection of y on γ_θ (see [6]) and remark that for $z \in (\pi(y), \theta)$, $G(o,z) = H(o,\pi(y))G(\pi(y),z)$ and $G(y,z) = H(y,\pi(y))G(\pi(y),z)$ by formula 1. Hence $\frac{G(y,z)}{G(o,z)} = \frac{H(y,\pi(y))}{H(o,\pi(y))}$ does not depend anymore on z and its limit when z goes to θ is then $K_\theta(y) = \frac{H(y,\pi(y))}{H(o,\pi(y))}$. By formula 1,

$$G(o,y)K_\theta(y) = H(y,\pi(y))\frac{G(o,y)}{H(o,\pi(y))} = H(y,\pi(y))H(\pi(y),y)G(y,y).$$

But $G(y,y) \geq p_2(y,y) \geq 3\varepsilon^2$ and $H(y,\pi(y))H(\pi(y),y) \geq \varepsilon^{2c}$ by $(\mathcal{H})$ and $d(y,\pi(y)) \leq c$, which finishes the proof of lemma 4.2.

Let us prove lemma 4.3 :

$$\begin{aligned} G_U(o,y) &= G(o,y) - E_o[G(X_\tau,y)\mathbf{1}_{(\tau<+\infty)}] \\ &= G(o,y)\left(1 - E_o\left[\frac{G(X_\tau,y)}{G(o,y)}\mathbf{1}_{(\tau<+\infty)}\right]\right) \end{aligned}$$

and by definition of Martin's kernel, if we could switch the limit and expectation, by a conditioning formula [6],

$$\lim_{y\in\Gamma_c^\theta, y\to\theta} \frac{G_U(o,y)}{G(o,y)} = 1 - E_o[K_\theta(X_\tau)\mathbf{1}_{(\tau<+\infty)}] = P_o^\theta[\tau = +\infty].$$

We now justify that inversion by Lebesgue's theorem. The idea is to bound, when τ is finite, $\frac{G(X_\tau,y)}{G(o,y)}$ by a multiple of $K_\theta(X_\tau)$. We compare for that purpose $G(X_\tau,y)$ with $K_\theta(X_\tau)$. Denote again by π the projection function on γ_θ. We distinguish two cases

If $\pi(X_\tau) \in [o,\pi(y)]$, $\frac{G(X_\tau,y)}{K_\theta(X\tau)} = \frac{G(\pi(X_\tau),y)}{K_\theta(\pi(X\tau))} = \frac{G(o,y)}{K_\theta(o)} = G(o,y)$, by formula 1 and the remark that this formula also implies by definition of K_θ and by taking the limit that $K_\theta(X_\tau) = H(X_\tau,\pi(X_\tau))K_\theta(\pi(X_\tau))$ and $K_\theta(o) = H(o,\pi(X_\tau))K_\theta(\pi(X_\tau))$.

If $\pi(X_\tau) \notin [o,\pi(y)]$, again $\frac{G(X_\tau,y)}{K_\theta(X\tau)} = \frac{G(\pi(X_\tau),y)}{K_\theta(\pi(X\tau))}$. We also have, by definition and formula 1, $K_\theta(\pi(X_\tau)) = (H(o,\pi(X_\tau)))^{-1}$, hence the quotient above equals $H(o,\pi(X_\tau))G(\pi(X_\tau),y) = H(o,\pi(y))H(\pi(y),\pi(X_\tau))G(\pi(X_\tau),y)$. We know that G is bounded (see [7, 6]) and H is a probability, so it just remains to compare $H(o,\pi(y))$ with $G(o,y)$. But $\frac{H(o,\pi(y))}{G(o,y)} = (G(\pi(y),y))^{-1}$ and $\frac{1}{G}$ is bounded by $\frac{1}{3\varepsilon^2}$.

Merging the two cases gives a constant β such that $\frac{G(X_\tau,y)}{K_\theta(X\tau)} \leq \beta G(o,y)$, which enables to use Lebesgue's theorem and completes the proof of lemma 4.3.

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